DEVELOPMENT OF OBLIQUE WAVES IN A TWO-DIMENSIONAL SUBSONIC BOUNDARY LAYER

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In regard to a two-dimensional subsonic laminar boundary layer the customary view is that its stability against plane waves (Tollmien–Schlichting waves), propagating in the same direction as the mean-velocity vector (direct waves), must be investigated [1]. This follows from the Squire theorem, which states that in a study of the transient instability of a plane-parallel flow the problem for a wave propagating at an angle to the direction of the principal velocity (oblique wave) reduces to a two-dimensional problem with a lower Reynolds number. This suggests that the instability Reynolds number is determined directly from two-dimensional analysis [2, 3]. For practical applications, however, it is very important to know the increments of the waves. In particular, semi-empirical methods of calculating the transition Reynolds number, based on the determination of the amplification of the unstable waves, are used extensively [4, 5].

As a rule, in the theory of hydrodynamic instability and transition in the boundary layer the Squire theorem is interpreted much more broadly: Reynolds number is taken to mean the transition Reynolds number. It is also assumed that for a fixed Reynolds number the local increment of the oblique wave is smaller than that of the direct wave for the same Reynolds number, but this does not follow from the theorem.

Moreover, the statement of the theorem concerning the instability Reynolds number is not always valid, even for a two-dimensional incompressible laminar boundary layer on the assumption of plane-parallel conditions, since the transformation used by Squire is strictly valid only when the characteristics of the average flow are invariant under this transformation (self-similar boundary layers) and when the wave numbers of the disturbances are real (transient instability) or the ratio of the complex component of the wave number of the oblique wave in the direction of the velocity of the mean flow and the complex wave number of the equivalent direct wave is real (the Reynolds number of the equivalent wave should be real). This substantially limits the applicability of the Squire theorem to boundary layers and virtually makes it valueless in the problem of determining the transition. The theorem was previously shown to be inapplicable to compressible boundary layers [2].

Our aim here is to show that the assertion that oblique waves are "less dangerous" for the transition to the turbulent regime is wrong even for a two-dimensional incompressible isothermal boundary layer in the plane-parallel approximation. In other words, it is asserted that oblique waves can have lower instability Reynolds numbers, larger increments, and in the end, lower transition Reynolds numbers, than do direct waves.

Formulation of the Problem. We consider a two-dimensional subsonic laminar boundary layer of an incompressible liquid on an isothermal surface. As the mathematical model we take the Navier–Stokes equations and the continuity equation. We investigate the stability of this boundary layer against disturbances of small amplitude. The main flow in the approximation of a plane-parallel boundary layer is characterized by profile U(x, y) of the longitudinal component of the velocity vector. The transverse component of the velocity vector is assumed to be zero and the thickness of the boundary layer, constant [1, 2]. The initial equations must be linearized in the mean flow in order to study the instability. The solution of the linearized system of equations is sought in the form of two plane waves:

$$(u', v', w', p') = (f, \varphi, h, \pi) \exp [i(\alpha x + \beta z - \omega t)].$$

Here p', u', v', and w' are the disturbances of the pressure and the components of the velocity vector; π, f, φ, and h are their amplitudes; α and β are the wave numbers in the x and z directions; ω is the angular frequency; and t is the time.

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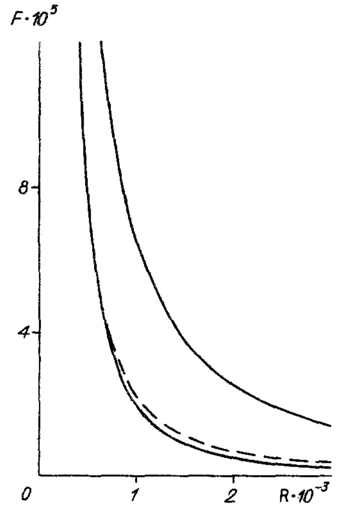

Fig. 1

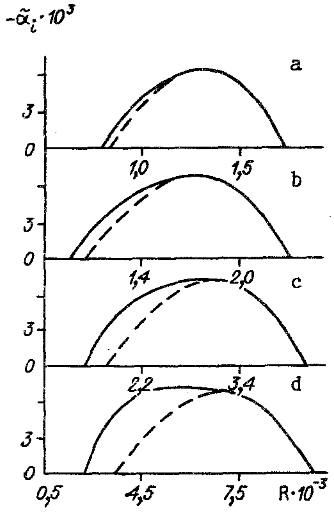

Fig. 2

Substituting these expressions into the linearized equations, we obtain [2]

$$i(\alpha U-\omega)f+U'\varphi=-i\alpha\pi+\frac{1}{R}\left[f''-\tilde{\alpha}^2 f\right],\ i(\alpha U-\omega)\varphi=$$
$$=-\pi'+\frac{1}{R}\left[\varphi''-\tilde{\alpha}^2\varphi\right],\ i(\alpha U-\omega)h=-i\beta\pi+\frac{1}{R}\left[h''-\tilde{\alpha}^2 h\right],\ i(\alpha f+\beta h)+\varphi'=0. \tag{1}$$

The boundary conditions are

$$f=0,\ \varphi=0,\ h=0\ (y=0),$$
$$f\to 0,\ \varphi\to 0,\ h\to 0\ (y\to\infty) \tag{2}$$

[$R = (U_e x/\nu)$ is the modified Reynolds number, calculated from the velocity at the outer boundary of the layer, and $\tilde{\alpha}^2 = \alpha^2 + \beta^2$].

Studying the stability thus comes down to finding the eigenvalues of the boundary-value problem (1), (2), i.e., the complex values of α and β as functions of the parameters R and ω (or the dimensionless frequency parameter $F = \omega\nu/U_e^2$). The boundary-value problem (1), (2) was solved numerically on a computer by an improved orthogonalized method [6, 7].

Results of Calculations. All the calculations were performed for an isothermal boundary layer on a flat plate. In Fig. 1 we show the neutral stability curves, constructed for direct and oblique waves. Here and below solid lines correspond to oblique waves and the dashed lines, to direct lines. The right branches of the curves for the two waves coincide. We see that in the region of rather low frequencies ($F \lesssim 10^4$) the minimum values of the instability Reynolds numbers for oblique waves of a fixed frequency become smaller than for the direct wave and the range of the unstable frequencies expands. This is particularly clear from the dependences of the local growth coefficients of the disturbances α_i (the imaginary parts of the wave number $\tilde{\alpha}$, for a plane wave $\tilde{\alpha}_i = \alpha_i$) on R, which are shown in Fig. 2, calculated for four values of the frequency parameter R: a) $F \cdot 10^5 = 3$, b) 2, c) 0.85, d) 0.25. It is interesting to note that for the first three frequencies the maximum local increment for an oblique wave coincides with the maximum for the direct wave, while it is larger for the fourth. This is because the maximum local increment for direct waves begins to decrease from a certain frequency as the frequency decreases further, and as a result the curves of the constant local increment for direct waves are closed.

The angle $\chi = \arctan(\beta/\alpha)$ between the mean-velocity vector and the direction of propagation of the wave is plotted against R in Fig. 3; χ is the angle at which the amplification of the wave is maximum. We see that near the left branch of the neutral curve this angle reaches its maximum value and then falls to zero at some value of R. It is smaller than the R at which the maximum value of $\tilde{\alpha}_i$ is reached for two values of the frequency parameter, is approximately equal to this value for the third and higher for the fourth.

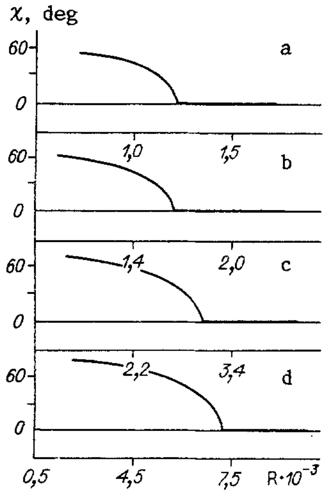

Fig. 3

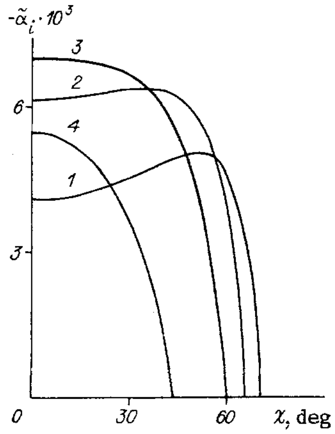

Fig. 4

For the frequency parameters $F \lesssim 0.85 \cdot 10^{-5}$ the wave with maximum amplification will be oblique on a segment greater than the region where the local amplification increment grows. The angle of inclination has its largest value at the beginning of this region, increases with decreasing F, and approaches 90° in the limit $F \to 0$.

In Fig. 4 we show the dependences of the local amplification coefficients on the angle of inclination χ of the wave, calculated for a fixed value $F = 2 \cdot 10^{-5}$ and $R \cdot 10^3 = 1.3$, 1.5, 1.7, and 2 (lines 1-4). From the graphs we can trace the evolution of these dependences as the oblique wave moves from the left branch of the neutral-stability curve to the right branch. We see, for example, that while for the first value of R, which is near the left branch, $\tilde{\alpha}_i$ is maximum for the oblique wave ($\chi = 55°$), for the second value of R it is approximately constant over a wide range $0 \leq \chi \leq 45°$. With further motion to the right branch the maximum shifts to the zero value of $\tilde{\alpha}_i$ and becomes more and more acute; this is consistent (only in this range of values of R) with the established ideas: the local increment is larger for the direct wave than for the oblique wave.

In practical applications, as a rule, the interest is not in the local increments but in their integral characteristics, which relate the disturbance amplitude A with its initial value A_0 by

$$J(x) = -\int_{x_1}^{x} \frac{\tilde{\alpha}_i}{\cos\chi} dx = \ln(A/A_0).$$

Here x_1 is the value of the coordinate x corresponding to the left branch of the neutral-stability curve. For a flat plate

$$J(\mathrm{R}) = -2\int_{\mathrm{R}_1}^{\mathrm{R}} \frac{\tilde{\alpha}_i}{\cos\chi} d\mathrm{R}, \mathrm{R}_1 = \left(\frac{U_e x_1}{\nu}\right)^{1/2}.$$

The integrated increments must also be known in order to determine the point of transition to the turbulent flow regime by the e^N method [4, 5], according to which a transition occurs when $J(x_2)$ [or $J(R_2)$, where $R_2 = (U_e x_2/\nu)^{1/2}$ is the value of R corresponding to the right branch of the neutral-stability curve] reaches a certain value N. For "full-scale" conditions the value N = 9 is usually reached. According to some data N reaches 11. Under the conditions of wind tunnel tests N may be substantially smaller [8].

The results of calculations of the integrated increments for four values of the frequency parameter are given in Fig. 5. We see that the values of J calculated for oblique waves can be much higher than the corresponding values for a direct wave; the differences become more pronounced when the dimensionless frequency parameter F is smaller. The ratio of the maximum values of J for direct and oblique waves at $F = 3 \cdot 10^{-5}$, $2 \cdot 10^{-5}$, $0.85 \cdot 10^{-5}$, and $0.25 \cdot 10^{-5}$ is 1.18, 1.25, 1.8, and 2.8 (a-d). And although in the first case this difference is not very large, we note that according to the e^N method the oblique wave leads

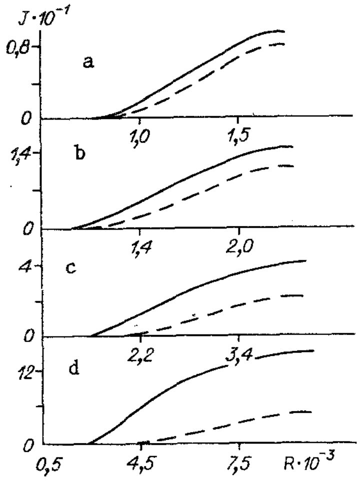

Fig. 5

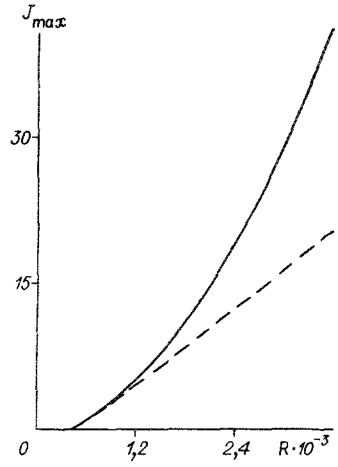

Fig. 6

to turbulence (N = 9) while this does not happen for the direct wave. In other words, oblique waves can be the cause of the transition to turbulence.

In Fig. 6 the maximum value J_{max} from the entire range of unstable frequencies is plotted as a function of R for direct and oblique waves. The excess of J_{max} for the oblique wave over its value for the direct wave becomes larger and larger as R increases. It also follows from Fig. 6 that oblique waves have a lower critical Reynolds number R_{cr} and cause a transition to turbulence at a point lying higher along the flow. For example, at N = 9 it is $R_{cr} = 1.6 \cdot 10^3$ as against $R_{cr} = 1.88 \cdot 10^3$ for the direct wave while at N = 11 it is $R_{cr} = 1.76 \cdot 10^3$ as against $R_{cr} = 2.09 \cdot 10^3$. This corresponds to the transition region being shortened by a factor of 1.38 in the first case and 1.41 in the second.

As for experimental investigations of oblique disturbances, we know of only one recent publication reporting on the first experimental study of artificially introduced controlled Tollmien–Schlichting waves on a flat isothermal plate in a wind tunnel [8]. A special technique was developed to generate such disturbances. A wave with dimensionless frequency parameter $F = 10^{-4}$ in the range of angles $0 \leq \chi \leq 25°$ was introduced into the boundary layer. The disturbances exhibited unusual behavior: the amplitude of the excited wave decreases as the angle of propagation increased, but amplification of the disturbances was recorded in the range of frequencies substantially smaller than the excited frequency. To explain this phenomenon Robey [8] suggests that a different, nonlinear mechanism amplifies oblique waves and that this mechanism is based on the fact that, as he demonstrates, the oblique wave differs qualitatively from the direct wave in that its vortex field has a three-dimensional structure.

In the light of our results, amplification of oblique waves can be explained even within the framework of the linear theory of hydrodynamic stability: the growing three-dimensional disturbances detected in the experiment are oblique Tollmien–Schlichting waves, which are unstable from the standpoint of the linear theory, while the excited wave is in the range of frequencies at which the direct wave is more stable than is the oblique wave.

In summary, we have demonstrated that oblique waves play a much greater role in the processes of the transition to turbulent flow than has been generally accepted. They can have lower instability Reynolds numbers and larger increments and can be the cause of turbulence. It is important to point out that this can occur not only because the critical transition Reynolds number can be substantially lower for oblique waves but also because they have a three-dimensional structure even in the linear theory.

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